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Qualitative Analysis of Stationary Generalized Viscoplastic Fluid Flows

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ABSTRACT

In this article, we consider the qualitative analysis of the generalized stationary regularized viscoplastic fluid equations with highly nonlinear viscosity. The existence and uniqueness of weak solutions are proved using the theory of monotone operators. Under certain conditions, the contractive property of the viscoplastic operator is used to be a tool to show the stability of solutions for the Dirichlet's problem with sources. Some technical properties are proved which have good fitting with the experimental results for some types of flows. A novel viscoplastic model is proposed to avoid the singularity of the nonlinear viscoplastic viscosity.

Keywords: Generalized viscoplastic fluid, Fixed point theorems, Monotone Operators.

1. Introduction

The viscoplastic fluid is a material which behaves like a rigid medium in a region if the external stress τ equals and not exceeding a certain critical value called the yield stress(the yield limit τ_s). This fluid in other region behaves like incompressible fluid if the external stress is exceeding the yield limit. The viscosity of this fluid depends mainly on the shear rate or the symmetric part of deformation tensor $\mathbf{D} = \frac{\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T}{2}$(where $\boldsymbol{u}$ is the velocity). The generalized viscoplastic fluid occurs in many applications for instance the geophysical as well as the industrial and hemodynamics applications, for comprehensive review, see Elborhamy (2012). The interested model for generalized viscoplastic fluid can be considered as a generalization for Herschel-Bulkley, Casson and Bingham models. The nonlinear generalized viscoplastic viscosity has the following form

$$\nu = \mu \|\mathbf{D}\|^{p-2} + \frac{\beta}{\|\mathbf{D}\|^{\alpha}} + \frac{\tau_s}{\|\mathbf{D}\|}, \tag{1}$$

where $\mu = \frac{1}{Re}$ is kinetic viscosity and $\|\mathbf{D}\|$ is the Frobenuis norm of the deformation tensor. β, α and p are constants with $\beta \geq 0$, $0 < \alpha < 1$ and $p \geq 2$. The famous viscoplastic viscosity models and the power law model can predicted from the generalized viscoplastic viscosity as follows:

- Bingham viscoplastic model, it is deduced with $\beta = 0$ and $p = 2$.
- Casson viscoplastic model, it is deduced with $\beta = \sqrt{\mu\tau_s}$, $p = 2$ and $\alpha = 0.5$.
- Herschel-Bulkley viscoplastic model, it is deduced with $\beta = 0$.
- Power law model, it is deduced with $\beta = 0$ and $\tau_s = 0$.
- Newtonian model, it is deduced with $\beta = 0$, $\tau_s = 0$ and p=2.

In the viscoplastic models, the most apparent difficulty is that the two regions are unknown a priori and finding them is a part of the problem, also the viscosity becomes singular in the rigid zones (plug or dead). A common way to avoid this difficulty is to use the regularization techniques. This can be done with different ways of regularizations (for comprehensive reviews about the regularization techniques and their properties, see Elborhamy (2012)). Therefore, whenever we need to regularize the generalized model, we will use the following simple form

$$\nu_\epsilon = \mu \|\mathbf{D}\|^{p-2} + \frac{\beta}{\|\mathbf{D}\|^{\alpha}} + \frac{\tau_s}{\|\mathbf{D}\|_\epsilon}, \tag{2}$$

where $\epsilon \in [0,1)$ is the regularizer and $\|\mathbf{D}(\boldsymbol{u})\|_\epsilon = (\epsilon^2 + \|\mathbf{D}\|^2)^{\frac{1}{2}}$.

In this work, we are studying the solution properties like the existence, the uniqueness, the regularity, the stability of solutions for stationary generalized viscoplastic fluid problem which is involved by the previous nonlinear viscosity model. We expose these properties of the solutions through the methods or the techniques which measure the behavior of the solution like fixed point techniques and the method of monotone operators for its the weak form. For the direct variational approach, the problem can not be dealt with it since the convective term creates a problem to have a zero potential function.

Additionally, the multi-valued calculus might be used to model the problem as a variational inequality from the second kind due to the non-differentiable term and its ability to describe the influence of the viscoplastic flow parameters on the flow behavior getting the similar description. To introduce the used methods, we need some abstract definitions involved in for the compactness, the coercivity, the continuity modes and the monotonicity modes. For a comprehensive and detailed reviews, see Evans (1998) and Roubíček (2010). To prove the existence and uniqueness for the continuous and the discretized problems, we will assume according to the standard argument in Girault and Raviart (1986) that the existence of the pressure π can be treated as a Lagrange multiplier.

In what follows, we use the standard notation for the functional spaces for the 2D vectors which are: for $1 \le p < \infty$ and $k > 0$, $\mathbf{L}^p(\boldsymbol{\Omega})$ and $\mathbf{W}^{k,p}(\boldsymbol{\Omega})$ are standard Lebesgue and Sobolev spaces. Also, $\mathbf{L}^p_0$ denotes the subspace of $\mathbf{L}^p$ of function of zero mean over $\boldsymbol{\Omega}$ and $\mathbf{W}^{1,p}_0(\boldsymbol{\Omega})$ is the space of functions in $\mathbf{W}^{1,p}_0(\boldsymbol{\Omega})$ with vanishing trace on $\partial\boldsymbol{\Omega}$. The subspace of $\mathbf{W}^{1,p}_0(\boldsymbol{\Omega})$ of divergence free vector functions is denoted by $\mathbf{V}$ and its dual space $\mathbf{V}^*$ and subspace of $L^2(\boldsymbol{\Omega})$ is denoted by Q . The norm in $\mathbf{W}^{1,p}_0$ is denoted by $\|.\|_{\mathbf{W}^{1,p}_0}$ and if p=2 then the norm is denoted by $\|.\|_1$. The norm in L^p is denoted by $\|.\|_p$ with $\|.\|_{L^2} = \|.\|_0$. We denote the best Sobolev constant by C_s where $\|\boldsymbol{u}\|_p \le C_s \|\mathbf{D}(\boldsymbol{u})\|_p$ for $\boldsymbol{u} \in \mathbf{W}^{1,p}_0(\boldsymbol{\Omega})$.

Let us introduce some technical results concerning the theory of nonlinear monotone operators.

Definition 1.1. *Let $A : V \to V^*$ be an operator on a space V. We say that the operator A is:*

- *Coercive iff* $lim_{\|\boldsymbol{u}\|\to\infty} \frac{\langle A\boldsymbol{u},\boldsymbol{u}\rangle}{\|\boldsymbol{u}\|} = \infty$;
- *Monotone iff* $\langle A\boldsymbol{u}_1 - A\boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2\rangle \geq 0$ *for all* $\boldsymbol{u}_1, \boldsymbol{u}_2 \in V$;
- *Strictly monotone iff* $\langle A\boldsymbol{u}_1 - A\boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2\rangle > 0$ *for all* $\boldsymbol{u}_1, \boldsymbol{u}_2 \in V$ *and* $\boldsymbol{u}_1 \neq \boldsymbol{u}_2$;
- *Strongly continuous if* $\boldsymbol{u}_n \to \boldsymbol{u}$ *implies* $A\boldsymbol{u}_n \to A\boldsymbol{u}$ *for all* $\boldsymbol{u}_n, \boldsymbol{u} \in V$;
- *Weakly continuous if* $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$ *implies* $A\boldsymbol{u}_n \rightharpoonup A\boldsymbol{u}$ *for all* $\boldsymbol{u}_n, \boldsymbol{u} \in V$;
- *Demicontinuous if* $\boldsymbol{u}_n \to \boldsymbol{u}$ *implies* $A\boldsymbol{u}_n \rightharpoonup A\boldsymbol{u}$ *for all* $\boldsymbol{u}_n, \boldsymbol{u} \in V$;
- *The operator A is said to be satisfying the M_0-condition if* $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$, $A\boldsymbol{u}_n \rightharpoonup f$ *and* $\langle A\boldsymbol{u}_n, \boldsymbol{u}_n\rangle \to \langle f, \boldsymbol{u}\rangle$ *imply* $A\boldsymbol{u} = f$.

Remark: In the sequel we denoted $C, C_0, C_1, ...$ various positive generic constants dependent of the discretization parameter or the regularization parameter and not necessarily the same.

In Section 2, we introduce the modeling of the stationary generalized viscoplastic fluid with its constitutive law. In Section 3, we introduce some technical results concerning the the generalized viscoplastic operator and Section 4, we prove the existence and uniqueness of solution using the method of monotone operator and fixed point techniques concluding the critical values for the viscoplastic flow parameters. In section 5 and 6, it is shown the condition to obtain the stable solution for the generalized viscoplastic equations and the well-posedness of the problem.

2. Modeling of Stationary Generalized Viscoplastic Fluid

Consider incompressible viscous flow in a bounded, simply connected domain $\boldsymbol{\Omega} \subset \mathbb{R}^2$ with Lipschitz continuous boundary $\partial\boldsymbol{\Omega}$. For the stationary flows the governing equations arising from the momentum and the continuity equations can be written as:

$$\boldsymbol{u}\boldsymbol{\nabla}\boldsymbol{u} = \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} + \boldsymbol{f}, \tag{3a}$$

$$\boldsymbol{\sigma} = -\pi\mathbf{I} + \boldsymbol{\tau}, \tag{3b}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \tag{3c}$$

$$\boldsymbol{\tau} = \begin{cases} 2\nu\mathbf{D}(u) & \text{if} \quad \|\mathbf{D}\| \neq 0 \\ \leq \boldsymbol{\tau}_s, & \text{if} \quad \|\mathbf{D}\| = 0, \end{cases} \tag{3d}$$

where $\boldsymbol{u}\boldsymbol{\nabla}\boldsymbol{u}$ is the convective term, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, π is the pressure, ν is the nonlinear viscosity and $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ satisfying the nonlinear growth condition.

Generally, these equations of motion can be casted in the following regularized form:

$$\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}\pi = \boldsymbol{\nabla}\cdot(2\nu_\epsilon\mathbf{D}) + \boldsymbol{f}, \tag{4a}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0. \tag{4b}$$

Boundary conditions complete the problem specification for the stationary flow problem, and with the Dirichlet's boundary data, we have

$$\boldsymbol{u} = \boldsymbol{u}^0 \quad \text{on} \quad \partial\boldsymbol{\Omega}. \tag{5}$$

So that, the generalized viscoplastic operator after scaling is

$$A\boldsymbol{u} = \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{\nabla}\cdot[(\mu\|\mathbf{D}\|^{p-2} + \frac{\beta}{\|\mathbf{D}\|^{\alpha}} + \frac{\tau_s}{\|\mathbf{D}\|})\mathbf{D}(\boldsymbol{u})],$$

and the regularized one is

$$A_\epsilon\boldsymbol{u} = \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{\nabla}\cdot[(\mu\|\mathbf{D}\|^{p-2} + \frac{\beta}{\|\mathbf{D}\|^{\alpha}} + \frac{\tau_s}{(\epsilon^2 + \|\mathbf{D}\|^2)^{\frac{1}{2}}})\mathbf{D}(\boldsymbol{u})].,$$

It can be written in the following mapping form: for the convective part, we have the mapping $A_c : \mathbf{V} \to \mathbf{V}^*$

$$A_c\boldsymbol{u} = \boldsymbol{u}.\boldsymbol{\nabla}\boldsymbol{u}, \tag{6}$$

and for the divergence part, we have the mapping $A_d : \mathbf{V} \to \mathbf{V}^*$

$$A_d\boldsymbol{u} = -div[(\mu\|\mathbf{D}\|^{p-2} + \frac{\beta}{\|\mathbf{D}\|^{\alpha}} + \frac{\tau_s}{\|\mathbf{D}\|})\mathbf{D}(\boldsymbol{u})] \;\; \forall\boldsymbol{u} \in \mathbf{W}_0^{1,p}, \tag{7}$$

with its regularized form

$$A_{d\epsilon}\boldsymbol{u} = -div[(\mu\left\|\mathbf{D}\right\|^{p-2} + \frac{\beta}{\left\|\mathbf{D}\right\|^{\alpha}} + \frac{\tau_s}{\left\|\mathbf{D}\right\|_{\epsilon}})\mathbf{D}(\boldsymbol{u})] \ \ \forall \boldsymbol{u} \in \mathbf{W}_0^{1,p}. \tag{8}$$

It can be written in a compact form with $A = A_c + A_d$ and $A_\epsilon = A_c + A_{d\epsilon}$ with

$$dom(A_\epsilon) = \{\boldsymbol{u} \in \mathbf{W}_0^{1,p}; A_\epsilon \boldsymbol{u} = 0, \boldsymbol{u}_{\partial\boldsymbol{\Omega}} = \boldsymbol{u}^0\}. \tag{9}$$

Let us introduce the following bilinear forms

$$\langle A_1\boldsymbol{u},\boldsymbol{v}\rangle = \int_{\boldsymbol{\Omega}} \mu\left\|\mathbf{D}\right\|^{p-2}\mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{v})d\boldsymbol{x}. \tag{10}$$

$$\langle A_2\boldsymbol{u},\boldsymbol{v}\rangle = \int_{\boldsymbol{\Omega}} \beta\frac{\mathbf{D}(\boldsymbol{u})}{\left\|\mathbf{D}\right\|^{\alpha}} : \mathbf{D}(\boldsymbol{v})d\boldsymbol{x}. \tag{11}$$

$$\langle A_3\boldsymbol{u},\boldsymbol{v}\rangle = \int_{\boldsymbol{\Omega}} \tau_s\frac{\mathbf{D}(\boldsymbol{u})}{\left\|\mathbf{D}\right\|_{\epsilon}} : \mathbf{D}(\boldsymbol{v})d\boldsymbol{x}. \tag{12}$$

$$b(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}) = \int_{\boldsymbol{\Omega}}(\boldsymbol{w}.\boldsymbol{\nabla u})\boldsymbol{v}d\boldsymbol{x} \leq C(\boldsymbol{\Omega})\left\|\boldsymbol{w}\right\|_p\left\|\boldsymbol{u}\right\|_p\left\|\boldsymbol{v}\right\|_p, \quad p \geq 1. \tag{13}$$

$$c(\pi,\boldsymbol{v}) = -\int_{\boldsymbol{\Omega}} \pi\boldsymbol{\nabla}\cdot\boldsymbol{v}d\boldsymbol{x}. \tag{14}$$

$$\langle A_\epsilon\boldsymbol{u},\boldsymbol{v}\rangle = \langle A_1\boldsymbol{u},\boldsymbol{v}\rangle + \langle A_2\boldsymbol{u},\boldsymbol{v}\rangle + \langle A_3\boldsymbol{u},\boldsymbol{v}\rangle. \tag{15}$$

The weak formulation of the generalized viscoplastic problem reads:

Find $\boldsymbol{u} \in \mathbf{V}$, $\pi \in Q$ *such that for any* $\boldsymbol{v} \in \mathbf{V}$ *and* $q \in Q$ *such that:*

$$B_\epsilon((\boldsymbol{u},\pi);(\boldsymbol{v},q)) = \langle A_\epsilon\boldsymbol{u},\boldsymbol{v}\rangle + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) + c(\pi,\boldsymbol{v}) + c(q,\boldsymbol{u}) = (\boldsymbol{f},\boldsymbol{v}). \tag{P}$$

Alternatively; The minimization problem corresponding to the stationary case with the viscosity specified by the conditions

$$0 < m_0 \leq \nu_\epsilon(\left\|\mathbf{D}\right\|) \leq M_0 < \infty, \quad \text{and} \quad 0 < m_1 \leq (\nu_\epsilon(\left\|\mathbf{D}\right\|)\left\|\mathbf{D}\right\|)' \leq M_1 < \infty. \tag{16}$$

can have

$$min\{J(\boldsymbol{v}); \boldsymbol{v} \in \mathbf{V}\}, \tag{17}$$

where,

$$J(\boldsymbol{v}) = \int_{\boldsymbol{\Omega}}\int_0^{\mathbf{D}(\boldsymbol{v})} 2\nu_\epsilon(z)zdz - \boldsymbol{f}\cdot\boldsymbol{v}, \tag{18}$$

and the corresponding weak statement is given by

$$(J'(\boldsymbol{u}), \boldsymbol{v}) = 0 \forall \boldsymbol{v} \in \mathbf{V}, \tag{19}$$

where J' is the Gateaux derivative of J with

$$J'(\boldsymbol{u}) = A\boldsymbol{u} - \boldsymbol{f} \tag{20}$$

As we shall see below, existence and uniqueness are straight forward consequences of the strict monotonicity and continuity property of the operator A_ϵ over the space $\mathbf{V}$ (see AGouzal (2005)).

3. Some Technical Results

Here, we prove some technical results using the inequalities of Korn and Friedrichs in Schweizer (2013) concerning the continuity, coercivity, boundedness and monotonicity of the bilinear form $\langle A_\epsilon \boldsymbol{u}, \boldsymbol{v}\rangle$ of the viscoplastic operator.

Proposition 3.1. *There exist constants C_1 and C_2 such that for $p \geq 2$*

$$\langle A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v}\rangle \geq C_1 \|u - v\|_1^p ; \tag{21}$$

and

$$\|A_1\boldsymbol{u} - A_1\boldsymbol{v}\|_{V^*} \leq C_2(\|\boldsymbol{u}\|_1 + \|\boldsymbol{v}\|_1)^{p-2}. \|\boldsymbol{u} - \boldsymbol{v}\|_1 \ \textit{for any } \boldsymbol{u}, \boldsymbol{v} \in \mathbf{H}_0^1. \tag{22}$$

Proof. It is in detailed in Wei (1992). □

Proposition 3.2. *The operator $A_\epsilon : \mathbf{V} \to \mathbf{V}^*$ is continuous.*

Proof. Let us assume that $\boldsymbol{u}_n \to \boldsymbol{u} \in \mathbf{V}(\boldsymbol{\Omega})$, then $\|\boldsymbol{u}_n - \boldsymbol{u}\|_{\mathbf{V}} \to 0$, and $\|\mathbf{D}(\boldsymbol{u}_n) - \mathbf{D}(\boldsymbol{u})\|_{\mathbf{L}^p} \to 0$ and hence using the convergence dominated theorem to obtain $\|A_\epsilon \boldsymbol{u}_n - A_\epsilon \boldsymbol{u}\|_{\mathbf{L}^p} \to 0$. One can easily to check that whenever $p \geq 2$,

$$\langle A_\epsilon \boldsymbol{u}, \boldsymbol{v}\rangle \leq \langle A_1 \boldsymbol{u}, \boldsymbol{v}\rangle + \langle A_2 \boldsymbol{u}, \boldsymbol{v}\rangle + \frac{\tau_s}{\epsilon} \|\boldsymbol{u}\|_1 \|\boldsymbol{v}\|_1 \ \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbf{V}. \tag{23}$$

□

Proposition 3.3. *The operator $A_\epsilon : \mathbf{V} \to \mathbf{V}^*$ is radially continuous or demi-continuous.*

Proof. For $\forall \boldsymbol{u}, \boldsymbol{v} \in \mathbf{V}$, the mapping $t \mapsto \langle A_\epsilon(\boldsymbol{u} + t\boldsymbol{v}, \boldsymbol{v}\rangle$ is continuous, then $\langle A_\epsilon \boldsymbol{u}, \boldsymbol{v}\rangle$ is radially continuous. For $\forall \boldsymbol{v} \in \mathbf{V}$ the functional $\boldsymbol{u} \mapsto \langle A_\epsilon \boldsymbol{u}, \boldsymbol{v}\rangle$ is continuous, then $\langle A_\epsilon \boldsymbol{u}, \boldsymbol{v}\rangle$ is demicontinuous. □

Proposition 3.4. *The operator* $A_\epsilon : \mathbf{V} \to \mathbf{V}^*$ *is coercive.*

Proof. It is easily to check that,

$$\langle A_\epsilon \boldsymbol{u}, \boldsymbol{u}\rangle \geq C_1 \|\boldsymbol{u}\|_{\mathbf{V}}^p \;\; \forall \boldsymbol{u} \in \mathbf{V}. \tag{24}$$

Then, we have $\frac{\langle A_\epsilon \boldsymbol{u}, \boldsymbol{u}\rangle}{\|\boldsymbol{u}\|_{\mathbf{V}}} \to \infty$ as $\|\boldsymbol{u}\|_{\mathbf{V}} \to 0$. □

Proposition 3.5. *The operator* $A_\epsilon : \mathbf{V} \to \mathbf{V}^*$ *is bounded.*

Proof. We prove $A_\epsilon(\{\boldsymbol{u} \in \mathbf{V}; \|\boldsymbol{u}\| \leq \rho\})$ bounded in $\mathbf{V}^*$ for any $\rho > 0$. Then, we have to estimate $\sup_{\|\boldsymbol{u}\| \leq \rho} \|A_\epsilon(\boldsymbol{u})\|_{\mathbf{V}^*}$, where:

$$\sup_{\|\boldsymbol{u}\| \leq \rho} \|A_\epsilon(\boldsymbol{u})\|_{\mathbf{V}^*} = \sup_{\|\boldsymbol{u}\| \leq \rho} \sup_{\|\boldsymbol{v}\| \leq 1} \langle A_\epsilon(\boldsymbol{u}), \boldsymbol{v}\rangle = \sup_{\|\boldsymbol{u}\| \leq \rho} \sup_{\|\boldsymbol{v}\| \leq 1} \int_{\boldsymbol{\Omega}} A_\epsilon(\boldsymbol{u}).\mathbf{D}(\boldsymbol{v}). \tag{25}$$

Then, we obtain

$$\sup_{\|\boldsymbol{u}\| \leq \rho} \sup_{\|\boldsymbol{v}\| \leq 1} \int_{\boldsymbol{\Omega}} A_\epsilon(\boldsymbol{u}).\mathbf{D}(\boldsymbol{v}) \leq \sup_{\|\boldsymbol{u}\| \leq \rho} \sup_{\|\boldsymbol{v}\| \leq 1} \|A_\epsilon(\boldsymbol{u})\|_{\mathbf{V}} \|\boldsymbol{v}\|_{\mathbf{V}} \leq \sup_{\|\boldsymbol{u}\| \leq \rho} \|A_\epsilon(\boldsymbol{u})\|_{\mathbf{V}}, \tag{26}$$

which means, it is bounded uniformly for ranging over a bounded set in $\mathbf{V}$. □

Proposition 3.6. *The operator* $A_\epsilon : \mathbf{V} \to \mathbf{V}^*$ *is uniformly monotone.*

Proof. For the convective term, it can deduce that $\int_{\boldsymbol{\Omega}} (\boldsymbol{z}.\boldsymbol{\nabla}\boldsymbol{u})\boldsymbol{v} = -\int_{\boldsymbol{\Omega}} (\boldsymbol{z}.\boldsymbol{\nabla}\boldsymbol{v})\boldsymbol{u} - \int_{\boldsymbol{\Omega}} (\boldsymbol{u}.\boldsymbol{v})(\boldsymbol{\nabla}.\boldsymbol{z}) + \int_{\partial\boldsymbol{\Omega}} (\boldsymbol{u}.\boldsymbol{v})\boldsymbol{z}.\boldsymbol{n}$ then $\int_{\boldsymbol{\Omega}} (\boldsymbol{z}.\boldsymbol{\nabla}\boldsymbol{u})\boldsymbol{v} = \frac{1}{2}\int_{\partial\boldsymbol{\Omega}} (\boldsymbol{u}.\boldsymbol{v})\boldsymbol{z}.\boldsymbol{n} \geq 0$ (with $\boldsymbol{\nabla}.\boldsymbol{z} = 0$ and $\boldsymbol{n}$ is the out-normal vector) which implies that the convection enjoys with the strict monotonicity. With respect to the $\langle A_\epsilon \boldsymbol{u}, \boldsymbol{v}\rangle$, it can be proven as follows:

$$\begin{aligned}\langle A_\epsilon \boldsymbol{u} - A_\epsilon \boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v}\rangle \geq 2\mu \|u - v\|_1^p + \textstyle\int_{\boldsymbol{\Omega}} \left(\frac{2\beta}{\|\mathbf{D}(\boldsymbol{u})\|^\alpha} + \frac{2\tau_s}{\|\mathbf{D}(\boldsymbol{u})\|_\epsilon}\right) \|\mathbf{D}(\boldsymbol{u}) - \mathbf{D}(\boldsymbol{v})\|^2 \\ - \textstyle\int_{\boldsymbol{\Omega}} \frac{2\beta}{\|\mathbf{D}(\boldsymbol{u})\|^\alpha} \frac{\|\mathbf{D}(\boldsymbol{u}) - \mathbf{D}(\boldsymbol{v})\|}{\|\mathbf{D}(\boldsymbol{v})\|^\alpha} \mathbf{D}(\boldsymbol{v}) : (\mathbf{D}(\boldsymbol{u}) - \mathbf{D}(\boldsymbol{v})) \\ - \textstyle\int_{\boldsymbol{\Omega}} \frac{2\tau_s}{\|\mathbf{D}(\boldsymbol{u})\|_\epsilon} \frac{\|\mathbf{D}(\boldsymbol{u}) - \mathbf{D}(\boldsymbol{v})\|}{\|\mathbf{D}(\boldsymbol{v})\|_\epsilon} \mathbf{D}(\boldsymbol{v}) : (\mathbf{D}(\boldsymbol{u}) - \mathbf{D}(\boldsymbol{v})).\end{aligned} \tag{27}$$

The terms $\frac{\|\mathbf{D}(\boldsymbol{v})\|}{\|\mathbf{D}(\boldsymbol{v})\|_\epsilon}$ and $\|\mathbf{D}(\boldsymbol{v})\|^{1-\alpha}$, are less than or equal to one. Consequently the term $\langle A_\epsilon \boldsymbol{u} - A_\epsilon \boldsymbol{v}, \boldsymbol{u}-\boldsymbol{v}\rangle$ should be positive, then we get

$$\langle A_\epsilon \boldsymbol{u} - A_\epsilon \boldsymbol{v}, \boldsymbol{u}-\boldsymbol{v}\rangle \geq C_1 \|u-v\|_1^p + C_2 \|\boldsymbol{u}-\boldsymbol{v}\|_1^2 \, \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbf{V}. \tag{28}$$

Then followed, the bilinear form $\langle A_\epsilon \boldsymbol{u}, \boldsymbol{v}\rangle$ is uniformly monotone. □

From that, we have the following at $p = 2$

$$\langle A_\epsilon \boldsymbol{u} - A_\epsilon \boldsymbol{v}, \boldsymbol{u}-\boldsymbol{v}\rangle \geq C(\boldsymbol{\Omega})(\mu + \beta + \frac{\tau_s}{\epsilon})^2 \|\boldsymbol{u}-\boldsymbol{v}\|_1^2, \tag{29}$$

and

$$\|A_\epsilon \boldsymbol{u} - A_\epsilon \boldsymbol{v}\| \leq C(\boldsymbol{\Omega})(\mu + \beta + \frac{\tau_s}{\epsilon}) \|\boldsymbol{u}-\boldsymbol{v}\|_1. \tag{30}$$

4. Existence and Uniqueness of the Weak Solutions

The existence of solution of the model problem (P) can be proved whenever the operator is additionally bounded and radially continuous. Generally, what we need to prove is the surjection of the operator A_ϵ which means, there exists at least one solution $\boldsymbol{u} \in \mathbf{V}(\subseteq \mathbf{W}_0^{1,p})$ to the problem (P). We outline the following theorem to move to the proof of the intended request. Boccardo and Dacorogna were shown in Boccardo and Dacorogna (1984) that, the monotonicity of $A_\epsilon(\boldsymbol{u})$ is necessary for the pseudo-monotonicity of the -$div(A_\epsilon(\boldsymbol{u}))$. With addition of the convective term we can prove the surjection of the two mapping by using the following theorem.

Theorem 4.1. *Let be $A_\epsilon = A_c + A_{d\epsilon} : \mathbf{V} \to \mathbf{V}^*$ be coercive and A_c is totaly continuous and $A_{d\epsilon}$ is radially continuous and monotone. Then is A_ϵ surjective.*

Proof. Define the Galerkin approximation $\boldsymbol{u}_k \in \mathbf{V}_k$ by the identity: $\forall \boldsymbol{v} \in \mathbf{V}_k$: $\langle A(\boldsymbol{u}_k), \boldsymbol{v}\rangle = \langle \boldsymbol{f}, \boldsymbol{v}\rangle$ we have $\langle A_c(\boldsymbol{u}_k) + A_{d\epsilon}(\boldsymbol{u}_k), \boldsymbol{v}\rangle = \langle \boldsymbol{f}, \boldsymbol{v}\rangle$. Choose a weakly convergent subsequence $\{\boldsymbol{u}_{k_i}\}_{i\in N}$ with a limit $\boldsymbol{u} \in \mathbf{V}$. From the monotonicity of $A_{d\epsilon}$, we get

$$0 \leq \langle A_{d\epsilon}(\boldsymbol{v}_l) - A_{d\epsilon}(\boldsymbol{u}_{k_i}), \boldsymbol{v}_l - \boldsymbol{u}_{k_i}\rangle = \langle A_{d\epsilon}(\boldsymbol{v}_l), \boldsymbol{v}_l - \boldsymbol{u}_{k_i}\rangle + \langle A_c(\boldsymbol{u}_{k_i}) - \boldsymbol{f}, \boldsymbol{v}_l - \boldsymbol{u}_{k_i}\rangle,$$

for any $\boldsymbol{v}_l \in \mathbf{V}_l$ with $l \leq k$. Passing to the limit with $i \to \infty$, it gives

$$0 \leq \langle A_{d\epsilon}(\boldsymbol{v}), \boldsymbol{v}-\boldsymbol{u}\rangle + \langle A_c(\boldsymbol{u}) - \boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u}\rangle.$$

Then by density of $\bigcup_k \mathbf{V}_k \in \mathbf{V}$, consider $\boldsymbol{v}_l \to \boldsymbol{v}$ for $\boldsymbol{v} \in \mathbf{V}$ arbitrary, since A_n is demicontinuous and pass to the limit $l \to \infty$ to write,

$$0 \leq \langle A_{d\epsilon}(\boldsymbol{v}), \boldsymbol{v} - \boldsymbol{u} \rangle + \langle A_c(\boldsymbol{u}) - \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u} \rangle,$$

finally by using Minty's trick in Minty (1963), it completes the proof to show that $A_{d\epsilon}(\boldsymbol{u}) + A_c(\boldsymbol{u}) = \boldsymbol{f}$. □

Theorem 4.2. *Under the conditions of the continuity, boundedness, the coercivity and the monotonicity of the mapping in the problem (P), then the problem has a weak solution in the sense of Brézis.*

Proof. For $A_\epsilon : \mathbf{V} \to \mathbf{V}^*$, it is proved that the mapping is radially continuous monotone which is pseudomonotone, with the coercivity this leads to the surjection of the mapping,

$$B_\epsilon((\boldsymbol{u}, \pi); (\boldsymbol{v}, q)) = (\boldsymbol{f}, \boldsymbol{v}), \tag{31}$$

in the sense of Brézis (1968). Moreover, for each solution of $\boldsymbol{u}$, there exists a unique $\pi \in \mathbf{L}_0^2(\boldsymbol{\Omega})$ such that the pair $\{\boldsymbol{u}, \pi\}$ is a solution of the problem(P). □

4.1 The Conditions for the Uniqueness of the Weak Solutions

Typically, there are three different distinct classes of such abstract theorems for the fixed point theory that will be interesting to prove the existence and uniqueness of this problem. Now, we use Banach's theorem (see Schweizer (2013) and Zeidler) under such constraint to obtain the contractive property for the viscoplastic operator to the existence and the uniqueness of weak solutions.

Theorem 4.3. *There exists a unique weak solution for the problem (P) in the Banach's sense with a strict contraction property under the constraint* $\frac{1}{\mu+\beta+\frac{\tau_s}{\epsilon}} < \frac{1}{\sqrt[p]{C(\boldsymbol{\Omega})\|\boldsymbol{f}\|_{-1}}}$ *where* $C(\boldsymbol{\Omega})$ *is the best possible constant.*

Proof. Let us apply Banach's fixed point theorem whenever $p = 2$ in the corresponding space $\mathbf{V} \subseteq \mathbf{H}_0^1(\boldsymbol{\Omega})$ with the norm $\|\boldsymbol{v}\|_1 = \|\boldsymbol{v}\|_{\mathbf{V}}$. Let us define $\boldsymbol{f} \in \mathbf{H}_0^{-1}$. Thus $\boldsymbol{w} \in \mathbf{V}$ satisfies

$$B_\epsilon((\boldsymbol{u}, \pi); (\boldsymbol{v}, q)) = (\boldsymbol{f}, \boldsymbol{v}), \tag{32}$$

for each $\boldsymbol{v} \in \mathbf{V}$ and $\boldsymbol{u}(0) = \boldsymbol{u}_0$. Let us define $A_\epsilon : \mathbf{V} \to \mathbf{V}^*$ by setting $A_\epsilon \boldsymbol{u} = \boldsymbol{w}$. Now, we need to prove our claim which is A_ϵ is strict contractive under the mentioned condition, to prove, choose $\boldsymbol{u}, \tilde{\boldsymbol{u}} \in \mathbf{V}$ and we define $\boldsymbol{w} = A_\epsilon \boldsymbol{u}$,

$\tilde{\boldsymbol{w}} = A_\epsilon \tilde{\boldsymbol{u}}$. Consequently $\boldsymbol{w}$ verifies Eq.(32) for $\boldsymbol{f}(\boldsymbol{u})$ and $\tilde{\boldsymbol{w}}$ satisfies similar identity for $\boldsymbol{f}(\tilde{\boldsymbol{u}})$. The operator $A_\epsilon \boldsymbol{u}$ is invertible for each $\boldsymbol{u} \in \mathbf{V}$. Moreover, $T(\boldsymbol{u}) = (A\boldsymbol{u})^{-1} \in \pounds(\mathbf{V}^*; \mathbf{V})$. Using the properties of $B_\epsilon((\boldsymbol{u}, \pi); (\boldsymbol{v}, q))$, the embedding $\mathbf{H}_0^1(\boldsymbol{\Omega}) \hookrightarrow \mathbf{L}^2(\boldsymbol{\Omega})$ and the Poincare's inequality, we can get

$$\|\boldsymbol{w} - \tilde{\boldsymbol{w}}\|_0 \leq \frac{C(\boldsymbol{\Omega})}{(\mu + \beta + \frac{\tau_s}{\epsilon})^2} \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_0 . \tag{33}$$

Next, it can be deduced that

$$[T(\boldsymbol{u}) - T(\boldsymbol{v})]\boldsymbol{f} = T(\boldsymbol{u})[A\boldsymbol{v} - A\boldsymbol{u}]T(\boldsymbol{v})\boldsymbol{f}. \tag{34}$$

We deduce the following depending on C,

$$\|[T(\boldsymbol{u}) - T(\boldsymbol{v})]\boldsymbol{f}\|_{\mathbf{V}} \leq \frac{C(\boldsymbol{\Omega})}{(\mu + \beta + \frac{\tau_s}{\epsilon})^2} \|\boldsymbol{f}\|_{\mathbf{V}^*} \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_0 . \tag{35}$$

So, when $\frac{C(\boldsymbol{\Omega})}{(\mu+\beta+\frac{\tau_s}{\epsilon})^2} \|\boldsymbol{f}\|_{-1} < 1$ is small enough then A_ϵ will be strict contraction. Then, we deduce that in the space $\mathbf{V}$, A_ϵ has a fixed point $\boldsymbol{u} \in \mathbf{V}$. However, for any $\frac{1}{\mu+\beta+\frac{\tau_s}{\epsilon}} < \frac{1}{\sqrt{C(\boldsymbol{\Omega})\|\boldsymbol{f}\|_{-1}}}$, the Banach's fixed point theorem is applied to obtain a conditional weak solution $\boldsymbol{u}$ for the problem (P). This condition is function dependent on the properties of the viscoplastic fluid, e.g the kinematic viscosity, the yield stress and the domain configuration. This result is proved numerically under the different values for Re and τ_s for configuring domains, e.g., the flow in driven cavity and the flow around a circular cylinder in Elborhamy (2012). This proof can be extended for $p > 2$ and the constraint can be taken in to the account but with an additional parameter p to have the following form

$$\frac{1}{\mu + \beta + \frac{\tau_s}{\epsilon}} < \frac{1}{\sqrt[p]{C(\boldsymbol{\Omega}) \|\boldsymbol{f}\|_{-1}}}. \quad \square$$

$\square$

4.2 The Critical Values for the Flow Parameters

In Newtonian case we can obtain the critical value for the Reynolds number to have following form

$$Re_c^2 < \frac{1}{C(\boldsymbol{\Omega}) \|\boldsymbol{f}\|_{-1}}. \tag{36}$$

At low flow rates (i.e. small Reynolds numbers), steady laminar flow patterns are observed. If the flow rate is gradually increased, then a critical

Reynolds number will be reached when the steady flow loses stability; i.e; mathematically, this behavior is realized by a Hopf bifurcation of the Navier-Stokes equations. This corresponds to the existence of a critical Reynolds number Re_c at which a single conjugate pair of eigenvalues of the linearized stability problem would cross the imaginary axis if the Reynolds number were further increased. A second, very different, form of bifurcation occurs if linearized stability is lost at a critical Reynolds number Re, with a real eigenvalue moving from the left half to the right half-plane as the Reynolds number is increased. For $Re > Re_c$ the problem can have multiple steady solutions, some of which are stable and some which are not.

In Bingham flow, the critical value can be deduced from its variational inequality to have the following form

$$\tau_{s_c} \int_{\mathbf{\Omega}} \|D(\boldsymbol{v})\| \, d\boldsymbol{x} = \int_{\mathbf{\Omega}} \boldsymbol{f} \cdot \boldsymbol{v} d\boldsymbol{x}. \tag{37}$$

At small values of yield stress, two regions are observed (i.e. steady laminar flow and plug flow patterns). If the yield stress is gradually increased, then the flow domain is going to be blocked, mathematically, this behavior is realized to have two steady solutions, both of which are stable and the velocity is either null or constant.

To deduce the critical value of the regularization parameter, we can expand the viscoplastic Bingham term and after some manipulations we have the following form

$$\frac{1}{\mu_c + \frac{\tau_{s_c}}{\epsilon_c}} < \frac{1}{\sqrt{C(\mathbf{\Omega}) \, \|\boldsymbol{f}\|_{-1}}}. \tag{38}$$

However, generally the uniqueness of solution for the generalized viscoplastic fluid which maintains its stability can be formed in the following

$$\frac{\epsilon_c}{\epsilon_c \mu_c + \epsilon_c \beta_c + \tau_{s_c}} < \frac{1}{\sqrt[p]{C(\mathbf{\Omega}) \, \|\boldsymbol{f}\|_{-1}}}. \tag{39}$$

So, we might conclude that, if the critical value of the regularization parameter is gradually decreased, then the steady flow gains its stability; i.e; the existence of the Hopf bifurcation might be hardly existed certainly in the plug flow regimes. Then, the problem might have one steady solution, which can considered as a uniformly strongly stable solution.

Even if with a rather small of yield stress the decreasing of the regularization parameter has the same doing for the decreasing of Reynolds number on the stability of the solution. The critical generalized viscoplastic equation (39) can be reduced after neglecting the terms $\epsilon_c\mu_c + \epsilon_c\beta_c$ which is roughly zero at the lower values of the regularization parameter to have the following form

$$\frac{\epsilon_c}{\tau_{s_c}} < \frac{1}{\sqrt[p]{C(\boldsymbol{\Omega})\, \|\boldsymbol{f}\|_{-1}}}. \tag{40}$$

So, the ratio $\frac{\epsilon}{\tau_s}$ might be considered as controllable parameter for the existence of the bifurcation and the secondary vortices in the regularized viscoplastic problem. The viscoplastic problem can have the following linearized problem for the generalized model from the following governing equations,

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{\nabla}\cdot(\sum_{i=1}^{n}(-1)^i 2\mu_i)\,\|\mathbf{D}\|^i)\mathbf{D}(\boldsymbol{u}) - (\mu + \frac{\tau_s}{\epsilon})\Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{f}, \tag{41a}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0. \tag{41b}$$

However, the problem can be linearized to obtain by dropping the higher order terms of $\|\mathbf{D}\|$ around $\|\mathbf{D}\| \simeq 0$ the following,

$$\boldsymbol{u}\boldsymbol{\nabla}\boldsymbol{u} - \frac{\epsilon\mu + \tau_s}{\epsilon}\Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{f}, \tag{42a}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0. \tag{42b}$$

So, the new Reynolds number is $Re = \frac{\epsilon}{\epsilon\mu + \tau_s}$, consequently at $\epsilon,\ \tau_s,\ \mu \to 0$ then we can get $Re \to \infty$ following that the chaotic behavior of the solution which showed by the figure 1.

So that, the new proposed generalized viscoplastic problem may have the following corrected cast

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}\pi = \boldsymbol{\nabla}\cdot(2\nu_\epsilon\mathbf{D}) + \boldsymbol{f}(\boldsymbol{x},t), \quad \text{if} \quad \|\mathbf{D}\| \neq 0 \tag{43a}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \tag{43b}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla}\pi = \boldsymbol{f}(\boldsymbol{x},t) \quad \text{when} \quad \boldsymbol{u} = \boldsymbol{u}(t) \quad \text{if} \quad \|\mathbf{D}\| \simeq 0, \tag{43c}$$

$$\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_o(\boldsymbol{x}), \tag{43d}$$

$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{u}^o(t) \quad \text{on} \quad \partial\boldsymbol{\Omega}. \tag{43e}$$

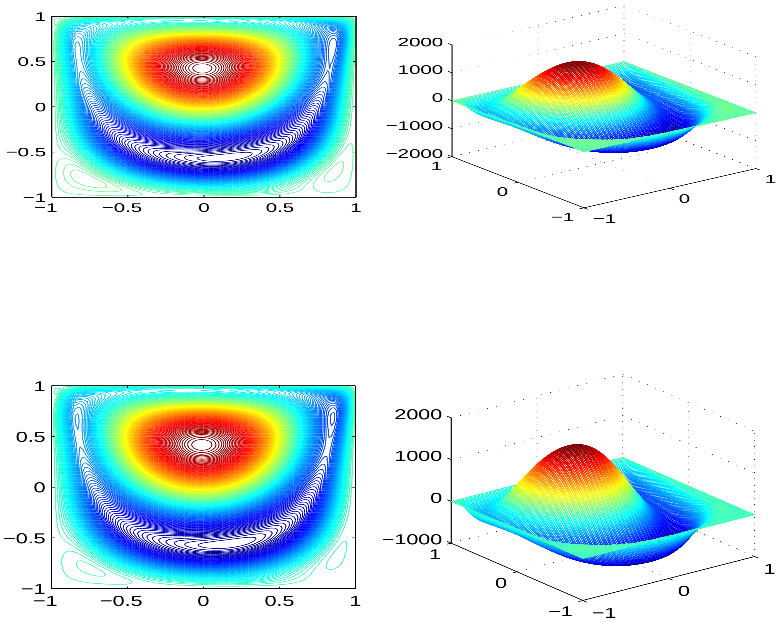

Figure 1: Simulation of stationary Casson fluid starting from up for $5E10^{-1}$ and $5E10^{-5}$ at $\mu = 75E10^{-5}$ respectively.

This result will be typically numerically proved in the next part or in the article.

5. Stability of Solutions

Theorem 5.1. *Assuming that the growth condition is satisfied. We assume that for all* $\boldsymbol{u} \in \mathbf{V}(\subseteq \mathbf{W}_0^{1,p})$, *there exists a subsequence* $\{k_i\}_{i=1}^{\infty}$ *such that* $\lim_{i\to\infty} \boldsymbol{f}_{k_i}(\boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ *weakly in* $\mathbf{L}^p(\boldsymbol{\Omega})$. *For each* k = *0, 1,2,... there exists a solution* $\boldsymbol{u}_k$ *to problem (P). There exists a subsequence* $\{\boldsymbol{u}_{k_n}\}_{n=1}^{\infty}$ *of the sequence* $\{u_k\}_{k=1}^{\infty}$ *and* $\bar{\boldsymbol{u}} \in \mathbf{V}$ *such that* $\boldsymbol{u}_{k_n} \to \bar{\boldsymbol{u}}$, *strongly in* $\mathbf{V}$.

$$\bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla} \bar{\boldsymbol{u}} = \boldsymbol{\nabla} \cdot (2\nu_\epsilon(\|\mathbf{D}(\bar{\boldsymbol{u}})\|)\mathbf{D}(\bar{\boldsymbol{u}})) + \boldsymbol{f}(\boldsymbol{x}, \bar{\boldsymbol{u}}), \tag{44a}$$

$$\bar{\boldsymbol{u}} = \boldsymbol{u}^0 \quad \text{on} \quad \partial\boldsymbol{\Omega}. \tag{44b}$$

Proof. First Step: $\boldsymbol{u}_{k_n}$ converges strongly to $\bar{\boldsymbol{u}}$ in $\mathbf{V}$

Assume that the linear growth condition holds. Then for all $k = 0, 1, 2,...$ there exists $\boldsymbol{u}_k \in \mathbf{V}_k$ such that

$$\boldsymbol{u}_k \cdot \boldsymbol{\nabla} \boldsymbol{u}_k = \boldsymbol{\nabla} \cdot (2\nu_\epsilon(\|\mathbf{D}(\boldsymbol{u}_k)\|)\mathbf{D}(\boldsymbol{u}_k)) + \boldsymbol{f}_k(\boldsymbol{x}, \boldsymbol{u}_k); \tag{45a}$$

$$\boldsymbol{u}_k = \boldsymbol{u}_k^0 \quad \text{on} \quad \partial\boldsymbol{\Omega}. \tag{45b}$$

It follows that for each $k = 0, 1, 2,...$ there exists $\boldsymbol{u}_k \in \mathbf{V}(\boldsymbol{\Omega})$ satisfying (P). Due to the fact that $\mathbf{V}_k \in \mathbf{V}$ it follows the sequence $\{\mathbf{D}(\boldsymbol{u}_k)\}_{k=1}^{\infty}$ is bounded in $\mathbf{L}^p(\boldsymbol{\Omega})$ and we may choose a weakly convergent subsequence in $\mathbf{V}$ which up to subsequence may be assumed to be strongly convergent in $\mathbf{L}^p(\boldsymbol{\Omega})$. We denote its limit by $\bar{\boldsymbol{u}}$. By the assumptions we may take a subsequence $\{k_i\}_{i=1}^{\infty}$ such that $lim_{i\to\infty}\boldsymbol{f}_{k_i}(\boldsymbol{x}, \bar{\boldsymbol{u}}) = \boldsymbol{f}(\boldsymbol{x}, \bar{\boldsymbol{u}})$ weakly in $\mathbf{L}^p(\boldsymbol{\Omega})$. From the growth condition, we get

$$ess \sup_{\boldsymbol{x}\in\boldsymbol{\Omega}} |\boldsymbol{f}_k(\boldsymbol{x}, \bar{\boldsymbol{u}}) - \boldsymbol{f}(\boldsymbol{x}, \bar{\boldsymbol{u}})| \leq C\bar{\boldsymbol{u}}. \tag{46}$$

Moreover using the above equation and the definition of $\mathbf{V}_k$, we obtain the following

$$\{\boldsymbol{u}_k \cdot \boldsymbol{\nabla}\boldsymbol{u}_k - \boldsymbol{\nabla} \cdot (2\nu_\epsilon(\|\mathbf{D}(\boldsymbol{u}_k)\|)\mathbf{D}(\boldsymbol{u}_k))\}_{k=1}^{\infty} \tag{47}$$

is weakly convergent in $\mathbf{L}^q(\boldsymbol{\Omega})$ and also

$$\{-\boldsymbol{\nabla} \cdot (2\nu_\epsilon(\|\mathbf{D}(\boldsymbol{u}_k)\|)\mathbf{D}(\boldsymbol{u}_k))\}_{k=1}^{\infty} \tag{48}$$

is weakly convergent in $\mathbf{L}^q(\boldsymbol{\Omega})$ up to a subsequence to a certain function $\boldsymbol{u} \in \mathbf{L}^p(\boldsymbol{\Omega})$. Thus, one can obtain that

$$\langle A_\epsilon \boldsymbol{u}_k - A_\epsilon \bar{\boldsymbol{u}}, \boldsymbol{u}_k - \bar{\boldsymbol{u}} \rangle \to 0. \tag{49}$$

Hence by the fact that the operator $A_\epsilon \boldsymbol{u}$ is continuous, bounded and strictly monotone having $\boldsymbol{u}_k \to \boldsymbol{u} \in \mathbf{V}(\boldsymbol{\Omega})$ provided $\boldsymbol{u}_k \rightharpoonup \boldsymbol{u} \in \mathbf{V}$ and $\lim_{k\to\infty} \sup\langle A_\epsilon \boldsymbol{u}_k - A_\epsilon \boldsymbol{u}, \boldsymbol{u}_k - \boldsymbol{u}\rangle \leq 0$. It follows by using the same argument in Galeswski (2007) that $\{\mathbf{D}(\boldsymbol{u}_k)\}_{k=1}^{\infty}$ is strongly convergent in $\mathbf{V}(\boldsymbol{\Omega})$ which leads to $\boldsymbol{u}_{k_n} \to \bar{\boldsymbol{u}}$.

Second Step: By Convexity of $\boldsymbol{f}_k$ we get for any $\boldsymbol{u} \in \mathbf{V}(\boldsymbol{\Omega})$

$$\int_{\boldsymbol{\Omega}} \langle \boldsymbol{f}_k(\boldsymbol{x}, \boldsymbol{u}_k) - \boldsymbol{f}_k(\boldsymbol{x}, \boldsymbol{u}), \boldsymbol{u}_k - \boldsymbol{u}\rangle d\boldsymbol{x} \geq 0. \tag{50}$$

Then it follows that

$$\int_{\boldsymbol{\Omega}} \langle A_\epsilon \boldsymbol{u}_k - \boldsymbol{f}_k(\boldsymbol{x}, \boldsymbol{u}), \boldsymbol{u}_k - \boldsymbol{u}\rangle \geq 0. \tag{51}$$

Since $\boldsymbol{u}_k \to \bar{\boldsymbol{u}}$ strongly in $\mathbf{L}^p(\boldsymbol{\Omega})$ and $\boldsymbol{f}_k(\boldsymbol{x},\boldsymbol{u}) \rightharpoonup \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})$ weakly in $\mathbf{L}^q(\boldsymbol{\Omega})$. So

$$\int_{\boldsymbol{\Omega}} \langle -\boldsymbol{f}_k(\boldsymbol{x},\boldsymbol{u}_k), \boldsymbol{u}_k - \boldsymbol{u}\rangle d\boldsymbol{x} \to \int_{\boldsymbol{\Omega}} \langle -\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}_k), \bar{\boldsymbol{u}} - \boldsymbol{u}\rangle d\boldsymbol{x} \tag{52}$$

Hence one can easily conclude that,

$$\int_{\boldsymbol{\Omega}} \langle A_\epsilon \bar{\boldsymbol{u}} - \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}), \bar{\boldsymbol{u}} - \boldsymbol{u}\rangle d\boldsymbol{x} \geq 0, \tag{53}$$

for any $\boldsymbol{u} \in \mathbf{V}(\boldsymbol{\Omega})$. Apply Minty's trick Minty (1963), i.e. we consider the points $\bar{\boldsymbol{u}} + t\boldsymbol{u}$ where $\boldsymbol{u} \in \mathbf{V}(\boldsymbol{\Omega})$, using the above inequality, one obtains

$$\int_{\boldsymbol{\Omega}} \langle A_\epsilon \bar{\boldsymbol{u}} - \boldsymbol{f}(\boldsymbol{x},\bar{\boldsymbol{u}} + t\boldsymbol{u}), \boldsymbol{u}\rangle d\boldsymbol{x} \leq 0, \tag{54}$$

Since function $t \mapsto \int_{\boldsymbol{\Omega}} \langle \boldsymbol{f}(\boldsymbol{x}, \bar{\boldsymbol{u}}+t\boldsymbol{u}, \boldsymbol{u})\rangle d\boldsymbol{x}$ is continuous at any sufficiently small t then we obtain

$$\lim_{t\to 0} \int_{\boldsymbol{\Omega}} \langle A_\epsilon \bar{\boldsymbol{u}} - \boldsymbol{f}(\boldsymbol{x},\bar{\boldsymbol{u}} + t\boldsymbol{u}), \boldsymbol{u}\rangle d\boldsymbol{x} = \int_{\boldsymbol{\Omega}} \langle A_\epsilon \bar{\boldsymbol{u}} - \boldsymbol{f}(\boldsymbol{x},\bar{\boldsymbol{u}}), \boldsymbol{u}\rangle d\boldsymbol{x} \leq 0. \tag{55}$$

Combining with Eq.(46) we obtain

$$A_\epsilon \bar{\boldsymbol{u}} = \boldsymbol{f}(\boldsymbol{x},\bar{\boldsymbol{u}}) \tag{56}$$

Hence,

$$\bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla} \bar{\boldsymbol{u}} = \boldsymbol{\nabla} \cdot (2\nu_\epsilon(\|\mathbf{D}(\bar{\boldsymbol{u}})\|)\mathbf{D}(\bar{\boldsymbol{u}})) + \boldsymbol{f}(\boldsymbol{x},\bar{\boldsymbol{u}}), \tag{57a}$$

$$\bar{\boldsymbol{u}} = \boldsymbol{u}^0 \quad \text{on} \quad \partial\boldsymbol{\Omega}. \tag{57b}$$

□

6. The Well-Posedness Results

Theorem 6.1. *The problem (P) has a unique solution* $\{\boldsymbol{u}, \pi\} \in \{\mathbf{V}(\subseteq \mathbf{W}_0^{1,p}(\boldsymbol{\Omega}))\times Q(\subseteq L_0^2(\boldsymbol{\Omega}))$ *satisfying the following estimate*

$$\|u\|_p \leq C\,\|\boldsymbol{f}\|_{-1}\,, \tag{58}$$

$$\|\pi\|_0 \leq C(\mu\,\|\boldsymbol{u}\|_p + \beta\,\|\boldsymbol{u}\|_{1-\alpha} + \frac{\tau_s}{\epsilon}\,\|\boldsymbol{u}\|_1 + \|\boldsymbol{f}\|_{-1}), \tag{59}$$

Proof. To prove the first estimate, we use the bilinear form in (P) restricted to the divergence- free $\mathbf{V}$ to have

$$\langle A_\epsilon \boldsymbol{u}, \boldsymbol{v}\rangle + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \tag{60}$$

with the existence and uniqueness of solution of the pressure as a Lagrange multiplier corresponding to the div-free constraint which can be shown by a standard argument in Aposporidis and Veneziani (2011) and Girault and Raviart (1986). Put $\boldsymbol{v} = \boldsymbol{u}$ and $q = \pi$ in the equation (P) with Eq.52 and applies the inequality $(\boldsymbol{f}, \boldsymbol{v}) = \|\boldsymbol{f}\|_{-1} \|\boldsymbol{v}\|_0$ to estimate the rhs, then we have

$$\|u\|_p \leq C \|\boldsymbol{f}\|_{-1}. \tag{61}$$

To prove the second estimate by using the Nečas's inequality, the bound for the pressure follows

$$\|\pi\|_0 \leq C \sup_{\boldsymbol{v}\in \mathbf{W}_0^{1,p}} \frac{\langle \boldsymbol{\nabla}\cdot \boldsymbol{v}, q\rangle}{\|\boldsymbol{v}\|_1}. \tag{62}$$

From the equation (P) using the Korn's and Cauchy's inequalities, one can obtain after putting q=0,

$$\frac{\langle \boldsymbol{\nabla}\cdot \boldsymbol{v}, \pi\rangle}{\|\boldsymbol{v}\|_1} = \mu \|\boldsymbol{u}\|_p + \beta \|\boldsymbol{u}\|_{1-\alpha} + \frac{\tau_s}{\epsilon} \|\boldsymbol{u}\|_1 + \|\boldsymbol{f}\|_{-1}. \tag{63}$$

Then the second estimate can be obtained as follows

$$\|\pi\|_0 \leq C(\mu \|\boldsymbol{u}\|_p + \beta \|\boldsymbol{u}\|_{1-\alpha} + \frac{\tau_s}{\epsilon} \|\boldsymbol{u}\|_1 + \|\boldsymbol{f}\|_{-1}). \tag{64}$$

One can conclude that, the nullity of the pressure space under the absence of the external force in the rigid and plug zones and this was proved numerically in Elborhamy (2012) and Grinevich and Olshanskii (2009). □

7. Conclusion

In this paper, we have presented the qualitative analysis for the generalized regularized stationary viscoplastic fluids problem in the continuous form. The coercivity and monotonicity of the viscoplastic operator are used to prove the existence and uniqueness of solution in the sense of Brézis. The contractive property of the operator is obtained under a certain conditions on the viscoplastic flow parameters, and it is led us to estimate their critical values. Generally, it is shown that the existence and stability of solution for a family of Dirichlet's problems with sources are strongly related to the flow parameters. Some analytical results for velocity and pressure are obtained to prove their well-posedness for most of the viscoplastic flow regimes. A novel viscoplastic model is proposed to avoid the troubles of regularization and the singularity of the viscoplastic viscosity.

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